CHAPTER 3

CERTAIN STRUCTURES AND CURVATURE TENSORS
INDUCED IN THE TANGENT BUNDLE

3.1 Introduction

Let $M$ be an $n$-dimensional differentiable manifold of class $C^\infty$ and $T(M)$ denotes tangent bundle of $M$. Then $T(M)$ is also a differentiable manifold of dimension $2n$ [5]. If the base space $M$, admits a tensor field $F$ of type $(1,1)$, a vector field $\xi$ and 1-form $\eta$ satisfying

\[(3.1.1)\]
\[\begin{align*}
(i) \quad & F^2 = a^r I_\eta + \eta \otimes \xi, \\
(ii) \quad & F\xi = 0 \\
(iii) \quad & \eta(\xi) = -a^r
\end{align*}\]

and for any complex number $a$ not zero, then we say that the base space $M$ admits Hsu- contact structure [4]. If $(1,1)$ tensor field $F$ on $M$ satisfies

\[(3.1.2)\]
\[F^2 = a^r I_n,
\]

Where $r \neq 0$, $r$ is a positive integer, we say that $M$ admits Hsu-structure. It is well known that $F^C$, $F^V$, $F^H$ denote complete, vertical and horizontal lifts of $F$ in $T(M)$. If we define

\[(3.1.3)\]
\[P = F^C + \frac{1}{a^{r/2}} \eta^V \otimes \xi^V + \frac{1}{a^{r/2}} \eta^C \otimes \xi^C
\]

and
Then it is easy to prove that $P$ and $Q$ define, Hsu-structure on $T(M)$ [2].

If $G$ be Riemannian metric on $M$ then $G^C$ given by

$$G^C (X^C,Y^C) = G(X,Y)^C$$

for each $X,Y \in \mathfrak{X}_0(M)$ defines the Riemannian metric in $T(M)$.

### 3.2 Induced Structures in $T(M)$

In this section we shall prove the following theorems.

**Theorem 3.2.1.** If $F$ gives a Hsu structure on the base space $M$ then (1,1) tensor field $K$ given by

$$K = F^C + \left( \frac{2^2+1}{a^2} \right) \eta^V \otimes \xi^V + \frac{\beta}{a^2} \eta^C \otimes \xi^H$$

defines the similar structure in $T(M)$, when $\beta, \gamma \in R, \quad \gamma \neq 0$.

**Proof.** Since $F^C \xi^C = F^C \xi^V = F^C \xi^H = 0$ [5].

hence proof follows in view of the definition of $K$ and the fact that $F^C$ admits Hsu-structure in $T(M)[5]$.

**Theorem 3.2.2.** For (1,1) tensor field $F$ admitting a Hsu-structure in $M$, the (1,1) tensor field $L$ given by

$$L = F^C + \left( \frac{2^2+1}{a^2} \right) \eta^V \otimes \xi^V + \frac{\beta}{a^2} \eta^V \otimes \xi^H$$

$$- \frac{\beta}{a^r/2} \eta^H \otimes \xi^V + \frac{\gamma}{a^r/2} \eta^H \otimes \xi^H$$

defines a Hsu-structure in $T(M)$

**Proof.** As $F^C \xi^V = F^C \xi^H = 0$ [5].
Hence the proof follows easily by virtue of the definition of $L$ and the fact that $F^C$ admits Hsu-structure in $T(M)$.

**Theorem 3.2.3.** If in $T(M)$, $(1,1)$ tensor field $J$ be defined as

$$JX^V = a^{r/2}X^H, \quad JX^H = a^{r/2}X^V$$

for each $X \in \mathcal{Z}_0^1(M)$, then $J$ defines a Hsu-structure in $T(M)$.

**Proof.** We have in view of equation (3.2.3)

$$J^2X^V = a^{r/2}JX^H = a^rX^V.$$ 

Hence, 

$$J^2 = a^rI_{2n}$$

and

$$J^2X^H = a^{r/2}JX^V = a^rX^H.$$ 

Therefore, 

$$J^2 = a^rI_{2n},$$

which proves the proposition.

### 3.3 Curvature Identities

Suppose the base space $M$ admits the Riemannian metric $G$ and the Riemannian connection $\nabla$. It is well known that $G^C$ given by

$$G^C(X^C, Y^C) = (G(X, Y))^C$$

defines the Riemannian metric and $\nabla^C$ given by

$$\nabla^C_{X^C}Y^C = (\nabla_X Y)^C$$

defines the Riemannian connection in $T(M)$.

If $R(X, Y)Z$ be the curvature tensor of $M$ with respect to connection $\nabla$, we have [3]

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$ 

Also the curvature tensor $R^C$ of connection $\nabla^C$ in $T(M)$ is given by

$$R^C(X^C, Y^C)Z^C = \nabla^C_{X^C} \nabla^C_{Y^C}Z^C - \nabla^C_{Y^C} \nabla^C_{X^C}Z^C$$

$$-\nabla^C_{[X^C,Y^C]}Z^C.$$ 

Now, we prove the following theorem:
**Theorem 3.3.1.** The Curvature tensor $R^C(X^C,Y^C)Z^C$ in $T(M)$ satisfies the following identity

$$R^C(X^C,Y^C)Z^C + R^C(Y^C,X^C)X^C + R^C(Z^C,X^C)Y^C = 0.$$ 

**Proof.** We have for the base space $M[3]$. 

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$ 

Similarly in $T(M)$, we have

$$R^C(X^C,Y^C)Z^C = \nabla^C_{X^C} \nabla^C_{Y^C} Z^C - \nabla^C_{Y^C} \nabla^C_{X^C} Z^C - \nabla^C_{[X^C,Y^C]} Z^C.$$ 

Interchanging $X^C, Y^C, Z^C$ in Cyclic order, we get

$$R^C(Y^C,Z^C)X^C = \nabla^C_{Y^C} \nabla^C_{Z^C} X^C - \nabla^C_{Z^C} \nabla^C_{Y^C} X^C - \nabla^C_{[Y^C,Z^C]} X^C$$

and

$$R^C(Z^C,X^C)Y^C = \nabla^C_{Z^C} \nabla^C_{X^C} Y^C - \nabla^C_{X^C} \nabla^C_{Y^C} Y^C - \nabla^C_{[Z^C,X^C]} Y^C.$$ 

Adding all the three equations, we have

$$R^C(X^C,Y^C)Z^C + R^C(Y^C,X^C)X^C + R^C(Z^C,X^C)Y^C$$

$$= \nabla^C_{X^C} \{ \nabla^C_{Y^C} Z^C - \nabla^C_{Z^C} Y^C \} + \nabla^C_{Y^C} \{ \nabla^C_{Z^C} X^C - \nabla^C_{X^C} Z^C \}$$

$$+ \nabla^C_{Z^C} \{ \nabla^C_{X^C} Y^C - \nabla^C_{Y^C} X^C \} - \nabla^C_{[X^C,Y^C]} X^C$$

$$- \nabla^C_{[Z^C,X^C]} Y^C - \nabla^C_{[X^C,Y^C]} Z^C.$$ 

Since, $\nabla^C_{X^C} Y^C - \nabla^C_{Y^C} X^C = [X^C, Y^C]$ and $\nabla^C$ is Riemannian connection in $T(M)$, hence

$$R^C(X^C,Y^C)Z^C + R^C(Y^C,X^C)X^C + R^C(Z^C,X^C)Y^C$$

$$= \{ \nabla^C_{X^C} [Y^C,Z^C] - \nabla^C_{Y^C} [Z^C,X^C] \} + \{ \nabla^C_{Y^C} [Z^C,X^C] \} - \nabla^C_{[Z^C,X^C]} Y^C + \{ \nabla^C_{Z^C} [X^C,Y^C] - \nabla^C_{[X^C,Y^C]} Z^C \}$$


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\[ = 0. \]

Hence the theorem is proved.

**Theorem 3.3.2.** If in the base space \( M \) for each \( X, Y \in \mathcal{S}_0^1(M) \) and for the Riemannian metric \( G \)

\[
K(X, Y) = \frac{G(X, R(X,Y)Y)}{G(X, X)G(Y, Y) - (G(X,Y))^2}
\]

then in \( T(M) \) we have

\[
K^C(X^C, Y^C) = K^C(sX^C, tY^C),
\]

where \( s, t \) are real numbers.

**Proof.** Since,

\[
K(X, Y) = \frac{G(X, R(X,Y)Y)}{G(X, X)G(Y, Y) - (G(X,Y))^2}.
\]

In \( T(M) \), we have in the same manner

\[
K(X^C, Y^C) = \frac{G(X^C, R(X^C,Y^C)Y^C)}{G^C(X^C, X^C)G^C(Y^C, Y^C) - (G^C(X^C,Y^C))^2}
\]

as hence it is show that

\[
K^C(sX^C, tY^C) = K^C(X^C, Y^C).
\]


